Existence of soliton solutions for a quasilinear Schrödinger equation involving critical exponent in $\mathbb{R}^N$

Abbas Moameni *

February 2, 2008

Department of Mathematics
University of British Columbia
Vancouver, B.C., Canada
moameni@math.ubc.ca

Abstract

Mountain pass in a suitable Orlicz space is employed to prove the existence of soliton solutions for a quasilinear Schrödinger equation involving critical exponent in $\mathbb{R}^N$. These equations contain strongly singular nonlinearities which include derivatives of the second order. Such equations have been studied as models of several physical phenomena. The nonlinearity here corresponds to the superfluid film equation in plasma physics.

Key words: Standing waves, quasilinear Schrödinger equations, critical exponent.
2000 Mathematics Subject Classification: 35J10, 35J20, 35J25.

1 Introduction

We study the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$i\partial_t z = -\epsilon\Delta z + W(x)z - l(|z|^2)z - k\epsilon\Delta h(|z|^2)h'(|z|^2)z, \quad x \in \mathbb{R}^N, N > 2, \tag{1}$$

*Research is supported by a Postdoctoral Fellowship at the University of British Columbia.

where $W(x)$ is a given potential, k is a real constant and l and h are real functions that are essentially pure power forms. Quasilinear equations of the form (1) have been established in several areas of physics corresponding to various types of h. The superfluid film equation in plasma physics has this structure for $h(s) = s$, (Kurihura in [7]). In the case $h(s) = (1+s)^{1/2}$, equation (1) models the self-channeling of a high-power ultra short laser in matter, see [19]. Equation (1) also appears in fluid mechanics [7,8], in the theory of Heidelberg ferromagnetism and magnus [9], in dissipative quantum mechanics [6] and in condensed matter theory [13]. We consider the case $h(s) = s$, $l(s) = \mu s^{\frac{p-1}{2}}$ and $k > 0$. Setting $z(t, x) = \exp(-iFt)u(x)$ one obtains a corresponding equation of elliptic type which has the formal variational structure:

$$- \epsilon\Delta u + V(x)u - \epsilon k(\Delta(|u|^2))u = \mu|u|^{p-1}u, \quad u > 0, x \in \mathbb{R}^N, \tag{2}$$

where $V(x) = W(x) - F$ is the new potential function.

Note that $p + 1 = 22^* = \frac{4N}{N-2}$ behaves like a critical exponent for the above equation [12, Remark 3.13]. For the subcritical case $p + 1 < 22^*$ the existence of solutions for problem (2) was studied in [10, 11, 12, 14, 15, 16] and it was left open for the critical exponent case $p + 1 = 22^*$ [12; Remark 3.13]. In the present paper, the existence of solutions is proved for $p + 1 = 22^*$ whenever the potential function $V(x)$ satisfies some geometry conditions. It is well-known that for the semilinear case ($k = 0$), $p + 1 = 2^*$ is the critical exponent. In this case there are many results about the existence of solutions for the subcritical and the critical exponent (e.g. [1, 4, 5, 17, 20]).

In the case $k > 0$, for a family of parameter μ, the existence of a nonnegative solution is proved for $N = 1$ by Poppenberg, Schmitt and Wang in [16] and for $N \geq 2$ by Liu and Wang in [11]. In [12] Liu and Wang improved these results for any $\mu > 0$ by using a change of variables and treating the new problem in an Orlicz space. The author in [14], using the idea of the fibrering method, studied this problem in connection with the corresponding eigenvalue problem for the laplacian $-\Delta u = V(x)u$ and proved the existence of multiple solutions for problem (2). It is established in [10], the existence of both one-sign and nodal ground states of soliton type solutions by the Nehari method. They also established some regularity of the positive solutions.

In this paper, we assume that the potential function V is radial, that is $V(x) = V(|x|)$, and satisfies the following conditions:

There exist $0 < R_1 < r_1 < r_2 < R_2$ and $\alpha > 0$ such that

$$V(x) = 0, \qquad \forall x \in \Omega := \left\{x \in \mathbb{R}^N : r_1 < |x| < r_2\right\}, \tag{A_1}$$

$$V(x) \geq \alpha, \qquad \forall x \in \Lambda^c, \tag{A_2}$$

where $\Lambda = \left\{x \in \mathbb{R}^N : R_1 < |x| < R_2\right\}$.
Here is our main Theorem.

Theorem 1.1. *There exists $\epsilon_0 > 0$, such that for all $\epsilon \in (0, \epsilon_0)$ problem (2) has a nonnegative solution $u_\epsilon \in H_r^1(\mathbb{R}^N)$ with $u_\epsilon^2 \in H_r^1(\mathbb{R}^N)$ and*

$$u_\epsilon(x) \longrightarrow 0 \quad as \quad |x| \longrightarrow +\infty.$$

This paper is organized as follows. In Section 2, we reformulate this problem in an appropriate Orlicz space. In Section 3, we prove the existence of a solution for a special deformation of problem (2). Theorem 1.1 is proved in Section 4.

2 Reformulation of the problem and preliminaries

Denote by $H_r^1(\mathbb{R}^N)$ the space of radially symmetric functions in

$$H^{1,2}(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : \bigtriangledown u \in L^2(\mathbb{R}^N)\right\}.$$

Denote by $C_{0,r}^\infty(\mathbb{R}^N)$ the space of radially symmetric functions in $C_0^\infty(\mathbb{R}^N)$ and by $D^{1,2}(\mathbb{R}^N)$ the following space,

$$D^{1,2}(\mathbb{R}^N) := \left\{u \in L^{2^*}(\mathbb{R}^N) : \bigtriangledown u \in L^2(\mathbb{R}^N)\right\}.$$

Without loss of generality, one can assume $k = 1$ in problem (2). We formally formulate problem (2) in a variational structure as follows

$$J_\epsilon(u) = \frac{\epsilon}{2}\int_{\mathbb{R}^N}(1+u^2)|\bigtriangledown u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx - \frac{1}{22^*}\int_{\mathbb{R}^N}|u|^{22^*}dx$$

on the space

$$X = \{u \in H_r^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty\},$$

which is equipped with the following norm,

$$\|u\|_X = \left\{\int_{\mathbb{R}^N}|\bigtriangledown u|^2 dx + \int_{\mathbb{R}^N} V(x)u^2 dx\right\}^{\frac{1}{2}}.$$

Liu and Wang in [12] for the subcritical case, i.e.

$$J_\epsilon(u) = \frac{\epsilon}{2}\int_{\mathbb{R}^N}(1+u^2)|\bigtriangledown u|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2dx-\frac{1}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}dx, \qquad (2<p+1<22^*),$$

by making a change of variables treated this problem in an Orlicz space. Following their work, we consider this problem for the critical exponent case ($p+1 = 22^*$) in the same Orlicz space. To convince the reader we briefly recall some of their notations and results that are useful in the sequel.

First, we make a change of variables as follows,

$$dv = \sqrt{1+u^2}du, \quad v = h(u) = \frac{1}{2}u\sqrt{1+u^2}+\frac{1}{2}\ln(u+\sqrt{1+u^2})$$

Since h is strictly monotone it has a well-defined inverse function: $u = f(v)$. Note that

$$h(u) \sim \begin{cases} u, & |u| \ll 1 \\ \frac{1}{2}u|u|, & |u| \gg 1, \quad h'(u) = \sqrt{1+u^2}, \end{cases}$$

and

$$f(v) \sim \begin{cases} v & |v| \ll 1 \\ \sqrt{\frac{2}{|v|}}v, & |v| \gg 1, \quad f'(v) = \frac{1}{h'(u)} = \frac{1}{\sqrt{1+u^2}} = \frac{1}{\sqrt{1+f^2(v)}}. \end{cases}$$

Also, for some $C_0 > 0$ it holds

$$G(v) = f(v)^2 \sim \begin{cases} v^2 & |v| \ll 1, \\ 2|v| & |v| \gg 1, \quad G(2v) \leq C_0G(v), \end{cases}$$

$G(v)$ is convex, $G'(v) = \frac{2f(v)}{\sqrt{1+f(v)^2}}$, $G''(v) = \frac{2}{(1+f(v)^2)^2} > 0$.

Using this change of variable, we can rewrite the functional $J_\epsilon(u)$ as

$$\bar{J}_\epsilon(v) = \frac{\epsilon}{2}\int_{\mathbb{R}^N}|\bigtriangledown v|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}V(x)f(v)^2dx - \frac{1}{22^*}\int_{\mathbb{R}^N}|f(v)|^{22^*}dx.$$

$\bar{J}_\epsilon$ is defined on the space

$$H^1_G(\mathbb{R}^N) = \{v|v(x) = v(|x|), \int_{\mathbb{R}^N}|\bigtriangledown v|^2dx < \infty, \int_{\mathbb{R}^N}V(x)G(v)dx < \infty\}.$$

We introduced the Orlicz space (e.g.[18])

$$E_G(\mathbb{R}^N) = \{v|\int_{\mathbb{R}^N}V(x)G(v)dx < \infty\}$$

equipped with the norm

$$|v|_G = \inf_{\zeta>0} \zeta(1+\int_{\mathbb{R}^N}(V(x)G(\zeta^{-1}v(x))dx),$$

and define the norm of $H^1_G(\mathbb{R}^N)$ by

$$\|v\|_{H^1_G(\mathbb{R}^N)} = |\bigtriangledown v|_{L^2(\mathbb{R}^N)} + |v|_G$$

Here are some related facts.

Proposition 2.1. *(i) $E_G(\mathbb{R}^N)$ is a Banach space.*

(ii) If $v_n \longrightarrow v$ in $E_G(\mathbb{R}^N)$, then $\int_{\mathbb{R}^N} V(x)|G(v_n)-G(v)|dx \longrightarrow 0$ and $\int_{\mathbb{R}^N} V(x)|f(v_n)-f(v)|^2dx \longrightarrow 0$.

(iii) If $v_n \longrightarrow v$ a.e. and $\int_{\mathbb{R}^N} V(x)G(v_n)dx \longrightarrow \int_{\mathbb{R}^N} V(x)G(v)dx$, then $v_n \longrightarrow v$ in $E_G(\mathbb{R}^N)$.

(iv) The dual space $E^*_G(\mathbb{R}^N) = L^\infty \cap L^2_V = \{w|w \in L^\infty, \int_{\mathbb{R}^N} V(x)w^2dx < \infty\}$.

(v) If $v \in E_G(\mathbb{R}^N)$, then $w = G'(v) = 2f(v)f'(v) \in E^*_G(\mathbb{R}^N)$, and $|w|_{E^*_G} = \sup_{|\phi|_{G\le 1}}(w,\phi) \le C_1(1+\int_{\mathbb{R}^N} V(x)G(v)dx)$, where C_1 is a constant independent of v.

(vi) For $N>2$ the map:$v \longrightarrow f(v)$ from $H^1_G(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ is continuous for $2 \le q \le 22^*$ and is compact for $2 < q < 22^*$.

(vii) Suppose B_k is the ball with center at the coordinate origin and radius $k>0$. Let $r<s$ and $Q = B_s \backslash B_r$. The map:$v \longrightarrow f(v)$ from $H^1_G(\mathbb{R}^N)$ into $L^q(Q)$ is compact for $q \ge 2$.

Proof. See Propositions (2.1) and (2.2) in [12] for the proof of parts (i) to (vi). We prove part (vii). Set $u = f(v)$. It is easy to check that $\|u\|_X \le \|v\|_{H^1_G(\mathbb{R}^N)}$. It is standard that the embedding from $H^1_r(\mathbb{R}^N)$ to $L^q(Q)$ is compact for $q \ge 2$ (e.g. [1]). Hence, we obtain the map:$v \longrightarrow f(v)$ from $H^1_G(\mathbb{R}^N)$ into $L^q(Q)$ is compact for $q \ge 2$. □

Hence forth, $\int, H^1, H^1_r, H^1_G, E_G, L^t$ and $\|\cdot\|$ stand for $\int_{\mathbb{R}^N}$, $H^{1,2}(\mathbb{R}^N)$, $H^1_r(\mathbb{R}^N)$, $H^1_G(\mathbb{R}^N)$, $E_G(\mathbb{R}^N)$, $L^t(\mathbb{R}^N)$ and $\|\cdot\|_{H^1_G(\mathbb{R}^N)}$ respectively. In the following we use C to denote any constant that is independent of the sequences considered.

3 Auxiliary Problem

In this section, we shall show some results needed to prove Theorem 1.1. Indeed, we first consider a special deformation $\bar{H}_\epsilon$ (See (3) in the following) of $\bar{J}_\epsilon$. Then, We show that the functional $\bar{H}_\epsilon$ satisfies all the properties of the Mountain Pass Theorem. Consequently, $\bar{H}_\epsilon$ has a critical point for each $\epsilon > 0$. We shall use this to prove Theorem 1.1 in the next section. In fact, we will see that the functionals $\bar{J}_\epsilon$ and $\bar{H}_\epsilon$ will coincide for the small values of ϵ. This idea was explored by Del Pino and Felmer [5].

To do this, we shall consider constants θ, k and β satisfying

$$4 < \theta < 22^*, \quad k > \frac{\theta}{\theta - 2}, \quad \beta = (\frac{\alpha}{k})^{\frac{1}{2(2^*-1)}}, \quad (\alpha \quad \text{is introduced in} \quad A_2)$$

and functions

$$\gamma(s) = \begin{cases} s^{22^*-1}, & s > 0, \\ 0, & s \leq 0 \end{cases}$$

$$\bar{\gamma}(s) = \begin{cases} \gamma(s), & s \leq \beta, \\ (\frac{\alpha}{k})s, & s > \beta, \end{cases}$$

$$w(x,s) = \chi_\Lambda(x)\gamma(s) + (1 - \chi_\Lambda(x))\bar{\gamma}(s),$$

where χ_Λ denotes the characteristic function of the set Λ. Set $W(x,t) = \int_0^t w(x,\zeta)d\zeta$. It is easily seen that the function w satisfies the following conditions,

$$0 \leq \theta W(x,t) \leq w(x,t)t, \quad \forall x \in \Lambda, t \geq 0, \qquad (g_1)$$

$$0 \leq 2W(x,t) \leq w(x,t)t \leq \tfrac{1}{k}V(x)t^2, \quad \forall x \in \Lambda^c, t \in \mathbb{R}. \qquad (g_2)$$

Now, we study the existence of solutions for the deformed equation, i.e.

$$-\epsilon\Delta u + V(x)u - \epsilon(\Delta(|u|^2))u = w(x,u), \quad x \in \mathbb{R}^N.$$

which correspond to the critical points of

$$H_\epsilon(u) = \frac{\epsilon}{2}\int(1+u^2)|\bigtriangledown u|^2 + \frac{1}{2}\int V(x)u^2 - \int W(x,u)dx.$$

As in Section (2), we can rewrite the functional $H_\epsilon(u)$ as a new functional $\bar{H}_\epsilon(v)$ with $u = f(v)$ as follows,

$$\bar{H}_\epsilon(v) = \frac{\epsilon}{2}\int|\bigtriangledown v|^2dx + \frac{1}{2}\int V(x)f(v)^2dx - \int W(x,f(v))dx. \qquad (3)$$

$\bar{H}_\epsilon(v)$ is defined on the Orlicz space H_1^G. To simplify the writing in this section, we shall assume $\epsilon = 1, H_1 = H$ and $\bar{H}_1 = \bar{H}$.
The following Proposition states some properties of the functional $\bar{H}$.

Proposition 3.1. *(i) $\bar{H}$ is well-defined on H_G^1.*

(ii) $\bar{H}$ is continuous in H_G^1.

(iii) $\bar{H}$ is Gauteaux-differentiable in H_G^1.

Proof. The proof is similar to the proof of Proposition (2.3) in [12] by some obvious changes.□

Here is the main result in this section.

Theorem 3.2. *$\bar{H}$ has a critical point in H_1^G, that is, there exists $0 \neq v \in H_1^G$ such that*

$$\int \bigtriangledown v. \bigtriangledown \phi dx + \int V(x)f(v)f'(v)\phi dx - \int w(x, f(v))f'(v)\phi dx = 0,$$

for every $\phi \in H_1^G$.

We use the Mountain Pass Theorem (see [2], [17]) to prove Theorem 3.2. First, let us define the Mountain Pass value,

$$C_0 := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \bar{H}(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], H_G^1) | \gamma(0) = 0, \bar{H}(\gamma(1)) \leq 0, \gamma(1) \neq 0\}.$$

The following Lemmas are crucial for the proof of Theorem 3.2.

Lemma 3.3. *The functional $\bar{H}$ satisfies the Mountain Pass Geometry.*

Proof. We need to show that there exists $0 \neq v \in H_G^1$ such that $\bar{H}(v) \leq 0$. Let $e \in C_{0,r}^\infty(\mathbb{R}^N)$ with $e \not\equiv 0$ and $\text{supp}(e) \subset \Omega$. It is easy to see that $H(te) \leq 0$ for the large values of t. Consequently $\bar{H}(v) < 0$ where $v = h(te)$. □

Lemma 3.4. *C_0 is positive.*

Proof. Set
$$S_\rho := \{v \in H^1_G | \int | \bigtriangledown v|^2 dx + \int V(x) f(v)^2 dx = \rho^2\}.$$
For $z = f(v)|f(v)|$ with $v \in S_\rho$, we have
$$\int | \bigtriangledown z|^2 dx = \int \frac{4f^2(v)}{1+f^2(v)} | \bigtriangledown v|^2 dx \le 4\rho^2$$
from which, we obtain
$$\begin{aligned} \int |f(v)|^{22^*} dx = \int |z|^{2^*} dx &\le C(\int | \bigtriangledown z|^2 dx)^{\frac{2^*}{2}} \\ &\le C\rho^{2^*}. \end{aligned} \tag{4}$$
Also, it follows from (g_1) and (g_2) that
$$\begin{aligned} \int W(x, f(v)) dx &= \int_\Lambda W(x, f(v)) dx + \int_{\Lambda^c} W(x, f(v)) dx \\ &\le \frac{1}{\theta} \int |f(v)|^{22^*} dx + \frac{1}{2k} \int V(x) f(v)^2 dx. \end{aligned} \tag{5}$$
Considering (4), (5) and the fact that $v \in S_\rho$, we obtain
$$\begin{aligned} \bar{H}(v) &= \frac{1}{2} \int | \bigtriangledown v|^2 dx + \frac{1}{2} \int V(x) f(v)^2 dx - \int W(x, f(v)) dx \\ &\ge \frac{1}{2}\rho^2 - C\rho^{2^*} - \frac{1}{2k}\rho^2 = (\frac{1}{2} - \frac{1}{2k})\rho^2 - C\rho^{2^*} \ge \frac{k-1}{4k}\rho^2, \end{aligned}$$
when $0 < \rho \le \rho_0 \ll 1$ for some ρ_0. Hence, for $v \in S_\rho$ with $0 < \rho \le \rho_0$ we have
$$\bar{H}(v) \ge \frac{k-1}{4k}\rho^2. \tag{6}$$
If $\gamma(1) = v$ and $\bar{H}(\gamma(1)) < 0$ then it follows from (6) that
$$\int | \bigtriangledown v|^2 dx + \int V(x) f(v)^2 dx > \rho_0^2,$$
thereby giving
$$\sup_{t \in [0,1]} \bar{H}(\gamma(t)) \ge \sup_{\gamma(t) \in S_{\rho_0}} \bar{H}(\gamma(t)) \ge \frac{k-1}{4k}\rho_0^2.$$
Therefore $C_0 \ge \frac{k-1}{4k}\rho_0^2 > 0.\square$

The Mountain Pass Theorem guaranties the existence of a $(PS)_{C_0}$ sequence $\{v_n\}$, that is, $\bar{H}(v_n) \longrightarrow C_0$ and $\bar{H}'(v_n) \longrightarrow 0$. The following Lemma states some properties of this sequence.

Lemma 3.5. *Suppose $\{v_n\}$ is a $(PS)_{C_0}$ sequence. The following statements hold.*

(i) $\{v_n\}$ *is bounded in* H^1_G.

(ii) *For each* $\delta > 0$, *there exists* $R > 4R_2$, *(*R_2 *is introduced in* (A_1) *and* (A_2)*) such that*

$$\limsup_{n\to+\infty} \int_{B_R^c} \big(|\bigtriangledown v_n|^2 + V(x)f(v_n)^2\big)dx < \delta.$$

(iii) *If* v_n *converges weakly to* v *in* H^1_G, *then*

$$\lim_{n\to+\infty} \int w(x, f(v_n))f(v_n)dx = \int w(x, f(v))f(v)dx.$$

(iv) *If* $v_n \geq 0$ *converges weakly to* v *in* H^1_G, *then for every nonnegative test function* $\phi \in H^1_G$ *we have*

$$\lim_{n\to+\infty} \langle \bar{H}'(v_n), \phi\rangle = \langle \bar{H}'(v), \phi\rangle.$$

Proof. Since $\{v_n\}$ is a $(PS)_{C_0}$ sequence, we have

$$\begin{aligned} \bar{H}(v_n) &= \frac{1}{2}\int |\bigtriangledown v_n|^2 dx + \frac{1}{2}\int V(x)f(v_n)^2 dx - \int W(x, f(v_n))dx \\ &= C_0 + o(1), \end{aligned} \tag{7}$$

and

$$\begin{aligned} \langle \bar{H}'(v_n), \phi\rangle &= \int \bigtriangledown v_n . \bigtriangledown \phi dx + \int V(x)f(v_n)f'(v_n)\phi dx - \int w(x, f(v_n))f'(v_n)\phi dx \\ &= o(\|\phi\|) \end{aligned} \tag{8}$$

For part (i), pick $\phi = \frac{f(v_n)}{f'(v_n)} = \sqrt{1 + f(v_n)^2}f(v_n)$ as a test function. One can easily deduce that $|\phi|_G \leq C|v_n|_G$ and

$$|\bigtriangledown \phi| = (1 + \frac{f(v_n)^2}{1 + f(v_n)^2})|\bigtriangledown v_n| \leq 2|\bigtriangledown v_n|,$$

which implies $\|\phi\| \leq C\|v_n\|$. Substituting ϕ in (8), gives

$$\begin{aligned} \langle \bar{H}'(v_n), \frac{f(v_n)}{f'(v_n)}\rangle &= \int (1 + \frac{f(v_n)^2}{1 + f(v_n)^2})|\bigtriangledown v_n|^2 dx + \int V(x)f(v_n)^2 dx \\ &\quad - \int w(x, f(v_n))f(v_n)dx \\ &= o(\|v_n\|). \end{aligned} \tag{9}$$

It follows from (g_1) and (g_2) that

$$-\int W(x, f(v_n))dx + \frac{1}{\theta}\int w(x, f(v_n))f(v_n)dx \geq \frac{1}{k}(\frac{1}{\theta} - \frac{1}{2})\int V(x)f(v_n)^2 dx \tag{10}$$

Taking into account (7), (9) and (10), we have

$$\begin{aligned}
C_0+o(1)+o(\|v_n\|) =&\bar{H}(v_n)-\frac{1}{\theta}\langle \bar{H}'(v_n),\frac{f(v_n)}{f'(v_n)}\rangle\\
=&\frac{1}{2}\int|\bigtriangledown v_n|^2dx+\frac{1}{2}\int V(x)f(v_n)^2dx-\int W(x,f(v_n))dx\\
&-\frac{1}{\theta}\int(1+\frac{f(v_n)^2}{1+f(v_n)^2})|\bigtriangledown v_n|^2dx-\frac{1}{\theta}\int V(x)f(v_n)^2dx\\
&+\frac{1}{\theta}\int w(x,f(v_n))f(v_n)dx\\
=&\int(\frac{1}{2}-\frac{1}{\theta}(1+\frac{f(v_n)^2}{1+f(v_n)^2}))|\bigtriangledown v_n|^2dx+(\frac{1}{2}-\frac{1}{\theta})\int V(x)f(v_n)^2dx\\
&+\int(\frac{1}{\theta}w(x,f(v_n))f(v_n)-W(x,f(v_n)))dx\\
\geq&(\frac{1}{2}-\frac{2}{\theta})\int|\bigtriangledown v_n|^2dx+(\frac{1}{2}-\frac{1}{\theta})(1-\frac{1}{k})\int V(x)f(v_n)^2dx.
\end{aligned}$$

Since, $\frac{1}{2}-\frac{2}{\theta}>0$ and $(\frac{1}{2}-\frac{1}{\theta})(1-\frac{1}{k})>0$ it follows from the above that $\int|\bigtriangledown v_n|^2dx+\int V(x)f(v_n)^2dx$ is bounded. It proves part (i).
For part (ii), let $\eta_R\in C^\infty(\mathbb{R}^N,\mathbb{R})$ be a function satisfying $\eta_R=0$ on $B_{\frac{R}{2}}$, $\eta_R=1$ on B_R^c and $|\bigtriangledown\eta_R(x)|\leq\frac{C}{R}$. It follows from part (i) that $\{v_n\}$ is bounded. Hence, from (8) we have

$$\langle \bar{H}'(v_n),\frac{f(v_n)}{f'(v_n)}\eta_R\rangle=o(1),$$

thereby giving

$$\begin{aligned}
&\int(1+\frac{f(v_n)^2}{1+f(v_n)^2})|\bigtriangledown v_n|^2\eta_Rdx+\int V(x)f(v_n)^2\eta_Rdx\\
&\qquad+\int\frac{f(v_n)}{f'(v_n)}\bigtriangledown v_n.\bigtriangledown\eta_Rdx=\int w(x,f(v_n))f(v_n)\eta_Rdx+o(1).
\end{aligned}$$

By (g_2), we get

$$w(x,f(v_n))f(v_n)\leq\frac{V(x)}{k}f(v_n)^2,\quad \forall x\in B^c_{\frac{R}{2}}.$$

Therefore,

$$\begin{aligned}
&\int(1+\frac{f(v_n)^2}{1+f(v_n)^2})|\bigtriangledown v_n|^2\eta_Rdx+\int(1-\frac{1}{k})V(x)f(v_n)^2\eta_Rdx\\
&\leq\frac{C}{R}\int\frac{|f(v_n)|}{f'(v_n)}|\bigtriangledown v_n|dx+o(1)\\
&\leq\frac{C}{R}\int|\bigtriangledown v_n|^2dx+\frac{C}{R}\int(|f(v_n)|^2+|f(v_n)|^4)dx+o(1). \qquad (11)
\end{aligned}$$

Also, it follows from part (vi) of Proposition 2.1 that $\{f(v_n)\}_n$ is a bounded sequence in $L^2(\mathbb{R}^N) \cap L^{22^*}(\mathbb{R}^N)$. Hence, $\int(|f(v_n)|^2 + |f(v_n)|^4)dx$ is bounded. Therefore, it follows from (11) that

$$\limsup_{n\to\infty} \int_{B_R^c} \big(| \bigtriangledown v_n|^2 dx + V(x) f(v_n)^2\big) dx < \delta, \quad (R > 4R_2).$$

It proves part (ii).

For part (iii), note first that from part (ii) of the present Lemma for each $\delta > 0$ there exists $R > 4R_2$ such that

$$\limsup_{n\to\infty} \int_{B_R^c} \big(| \bigtriangledown v_n|^2 + V(x) f(v_n)^2\big) dx < \frac{k\delta}{4}. \tag{12}$$

Since $B_R^c \subseteq \Lambda^c$, it follows from (g_2) that

$$w(x, f(v_n))f(v_n) \leq \frac{V(x)}{k} f(v_n)^2 \qquad \forall x \in B_R^c$$

which together with (12) imply that

$$\limsup_{n\to\infty} \int_{B_R^c} w(x, f(v_n))f(v_n) dx \leq \frac{\delta}{4}, \tag{13}$$

and consequently

$$\int_{B_R^c} w(x, f(v))f(v) dx \leq \frac{\delta}{4}.$$

It follows from (13) and the above inequality that

$$\begin{aligned} \Big| \int w(x, f(v_n))f(v_n) dx - \int w(x, f(v))f(v) dx \Big| \\ \leq \frac{\delta}{2} + \Big| \int_{B_{R_1}} \big[w(x, f(v_n))f(v_n) - w(x, f(v))f(v)\big] dx \Big| \\ + \Big| \int_{B_R \setminus B_{R_1}} \big[w(x, f(v_n))f(v_n) - w(x, f(v))f(v)\big] dx \Big|. \end{aligned} \tag{14}$$

Since $B_{R_1} \subset \Lambda^c$, we have

$$w(x, f(v_n))f(v_n) \leq \frac{V(x)}{k} f(v_n)^2, \quad \forall x \in B_{R_1}$$

Then, by the compact theorem embedding and Lebesgue Theorem, we obtain a subsequence still denoted by $\{v_n\}$, such that

$$\int_{B_{R_1}} w(x, f(v_n))f(v_n) dx \longrightarrow \int_{B_{R_1}} w(x, f(v))f(v) dx. \tag{15}$$

Also, it follows from part (vii) of Proposition 2.1 that the map $v \to f(v)$ from H^1_G into $L^q(B_R \backslash B_{R_1})$ is compact for every $q \geq 2$, hence

$$\int_{B_R \backslash \bar{B}_{R_1}} w(x, f(v_n)) f(v_n) dx \longrightarrow \int_{B_R \backslash \bar{B}_{R_1}} w(x, f(v)) f(v) dx. \tag{16}$$

Considering (15) and (16), it follows from (14) that

$$\limsup_{n \to \infty} \Big| \int w(x, f(v_n)) f(v_n) dx - \int w(x, f(v)) f(v) dx \Big| \leq \frac{\delta}{2},$$

for every $\delta > 0$. Consequently

$$\int w(x, f(v_n)) f(v_n) dx \longrightarrow \int w(x, f(v)) f(v) dx,$$

as $n \to \infty$. It proves part (iii).

To prove part (iv), note first that f is increasing and $f(0) = 0$, hence $f(v_n) \geq 0$ and $f(v) \geq 0$. For the second term on the right hand side of (8), we have

$$V(x) f(v_n) f'(v_n) \phi \leq V(x) f(v_n) \phi,$$

and since $v_n \rightharpoonup v$ weakly in H^G_1, for the right hand side of the above inequality we have

$$\lim_{n \to \infty} \int V(x) f(v_n) \phi \, dx = \int V(x) f(v) \phi \, dx.$$

Hence by the dominated convergence theorem and the fact that $v_n \to v$ a.e. we obtain

$$\lim_{n \to \infty} \int V(x) f(v_n) f'(v_n) \phi \, dx = \int V(x) f(v) f'(v) \phi \, dx. \tag{17}$$

For the third term on the right hand side of (8), we have

$$w(x, f(v_n)) f'(v_n) \phi \leq \frac{V(x)}{k} f(v_n) \phi, \qquad \forall x \in \Lambda^c,$$

and similarly by the dominated convergence theorem, we obtain

$$\lim_{n \to \infty} \int_{\Lambda^c} w(x, f(v_n)) f'(v_n) \phi \, dx = \int_{\Lambda^c} w(x, f(v)) f'(v) \phi \, dx. \tag{18}$$

Also, note that

$$w(x, f(v_n)) f'(v_n) \phi \leq f(v_n)^{22^* - 2} \phi \qquad \forall x \in \Lambda,$$

and from part (vii) of Proposition 2.1 that the map $v \to f(v)$ from H^1_G into $L^q(\Lambda)$ is compact for every $q \geq 2$, hence it follows again from the dominated convergence theorem that

$$\lim_{n \to \infty} \int_{\Lambda} w(x, f(v_n)) f'(v_n) \phi \, dx = \int_{\Lambda} w(x, f(v)) f'(v) \phi \, dx. \tag{19}$$

It follows from (8) and (17)-(19) that

$$\lim_{n\to+\infty}\langle \bar{H}'(v_n),\phi\rangle = \langle \bar{H}'(v),\phi\rangle.$$

It proves part (iv). $\square$

Lemma 3.6. *If $\{v_n\}$ is a $(PS)_{C_0}$ sequence, then v_n converges to $v\in H^1_G$. Consequently $\bar{H}(v)=\lim_{n\to+\infty}\bar{H}(v_n)$ and $\bar{H}'(v)=0$.*

Proof. It follows from part (i) of Lemma 3.5 that v_n is a bounded sequence in H^1_G. Hence, there exists $v\in H^1_G$ such that, up to a subsequence, $v_n\rightharpoonup v$ weakly in H^1_G and $v_n\to v$ a.e. in $\mathbb{R}^N$. Since we may replace v_n by $|v_n|$, we assume $v_n\geq 0$ and $v\geq 0$. Since, $\{v_n\}$ is a $(PS)_{C_0}$ sequence we have

$$\begin{aligned} o(\|v_n\|) &= \langle \bar{H}'(v_n),\frac{f(v_n)}{f'(v_n)}\rangle \\ &= \int (1+\frac{f(v_n)^2}{1+f(v_n)^2})|\bigtriangledown v_n|^2dx+\int V(x)f(v_n)^2dx \quad -\int w(x,f(v_n))f(v_n)dx \end{aligned} \tag{20}$$

and

$$o(\|v\|) = \langle \bar{H}'(v_n),\frac{f(v)}{f'(v)}\rangle. \tag{21}$$

It follows from part (iv) of Lemma 3.5 and (21) that

$$\begin{aligned} \langle \bar{H}'(v_n),\frac{f(v)}{f'(v)}\rangle &= \langle \bar{H}'(v),\frac{f(v)}{f'(v)}\rangle + o(\|v\|) \\ &= \int (1+\frac{f(v)^2}{1+f(v)^2})|\bigtriangledown v|^2dx+\int V(x)f(v)^2dx \\ &\quad -\int w(x,f(v))f(v)dx+o(\|v\|) \end{aligned} \tag{22}$$

In this step, we show that

$$\int \frac{f(v)^2|\bigtriangledown v|^2}{1+f(v)^2}\,dx \leq \liminf_{n\to\infty}\int \frac{f(v_n)^2|\bigtriangledown v_n|^2}{1+f(v_n)^2}\,dx.$$

Set $u_n=f(v_n)$ and $u=f(v)$. A direct computation shows that

$$\int |\bigtriangledown u_n^2|^2\,dx = 4\int \frac{f(v_n)^2|\bigtriangledown v_n|^2}{1+f(v_n)^2}\,dx \leq 4\|v_n\|^2.$$

Set $w_n=u_n^2$. It follows from the above that $\{w_n\}_n$ is a bounded sequence in $D^{1,2}(\mathbb{R}^N)$. Hence, up to a subsequence $w_n\rightharpoonup w$ weakly in $D^{1,2}(\mathbb{R}^N)$ and $w_n\to w$ a.e. in $\mathbb{R}^N$. It follows $w=u^2$. Also, by the lower semi continuity of the norm in $D^{1,2}(\mathbb{R}^N)$, we obtain

$$\int |\bigtriangledown w|^2\,dx \leq \liminf_{n\to\infty}\int |\bigtriangledown w_n|^2\,dx.$$

Plug $w_n = u_n^2$ and $w = u^2$ in this inequality to get

$$\int |\bigtriangledown u^2|^2\, dx \leq \liminf_{n\to\infty} \int |\bigtriangledown u_n^2|^2\, dx.$$

Substituting $u_n = f(v_n)$ and $u = f(v)$ in the above inequality gives

$$\int \frac{f(v)^2|\bigtriangledown v|^2}{1+f(v)^2}\, dx \leq \liminf_{n\to\infty} \int \frac{f(v_n)^2|\bigtriangledown v_n|^2}{1+f(v_n)^2}\, dx. \tag{23}$$

Also, lower semi continuity and Fatou's Lemma imply

$$\int |\bigtriangledown v|^2 dx \leq \liminf_{n\to\infty} \int |\bigtriangledown v_n|^2 dx, \tag{24}$$

$$\int V(x)G(v)dx \leq \liminf_{n\to\infty} \int V(x)G(v_n)dx. \tag{25}$$

Up to a subsequence one can assume

$$\liminf_{n\to\infty} \int |\bigtriangledown v_n|^2 dx = \lim_{n\to\infty} \int |\bigtriangledown v_n|^2 dx \tag{26}$$

$$\liminf_{n\to\infty} \int V(x)G(v_n)dx = \lim_{n\to\infty} \int V(x)G(v_n)dx. \tag{27}$$

$$\liminf_{n\to\infty} \int \frac{f(v_n)^2|\bigtriangledown v_n|^2}{1+f(v_n)^2}\, dx = \lim_{n\to\infty} \int \frac{f(v_n)^2|\bigtriangledown v_n|^2}{1+f(v_n)^2}\, dx. \tag{28}$$

It follows from (23)-(28) that there exist nonnegative numbers δ_1, δ_2 and δ_3 such that

$$\lim_{n\to\infty} \int |\bigtriangledown v_n|^2 dx = \int |\bigtriangledown v|^2 dx + \delta_1 \tag{29}$$

$$\lim_{n\to\infty} \int V(x)G(v_n)dx = \int V(x)G(v)dx + \delta_2. \tag{30}$$

$$\lim_{n\to\infty} \int \frac{f(v_n)^2|\bigtriangledown v_n|^2}{1+f(v_n)^2}\, dx = \int \frac{f(v)^2|\bigtriangledown v|^2}{1+f(v)^2}\, dx + \delta_3. \tag{31}$$

Now, we show that $\delta_1 = \delta_2 = \delta_3 = 0$. It follows from part (iii) of Lemma 3.5 that

$$\int w(x, f(v_n))f(v_n)dx \longrightarrow \int w(x, f(v))f(v)dx.$$

which together with (20) and (22) imply

$$\begin{aligned}\lim_{n\to\infty} \Big\{ \int (1+\frac{f(v_n)^2}{1+f(v_n)^2})|\bigtriangledown v_n|^2 dx \\ + \int V(x)f(v_n)^2 dx\Big\} = &\quad \lim_{n\to\infty} \int w(x, f(v_n))f(v_n)dx \\ &= \int w(x, f(v))f(v)dx \\ &= \int (1+\frac{f(v)^2}{1+f(v)^2})|\bigtriangledown v|^2 dx + \int V(x)f(v)^2 dx\end{aligned}$$

Taking into account (29), (30) and (31) the above limit implies $\delta_1 = \delta_2 = \delta_3 = 0$. Therefore, it follows from (29) and (30) that

$$\int |\bigtriangledown v|^2 dx = \lim_{n\to\infty} \int |\bigtriangledown v_n|^2 dx$$
$$\int V(x)G(v)dx = \lim_{n\to\infty} \int V(x)G(v_n)dx.$$

By Proposition 2.1, $v_n \longrightarrow v$ in E_G and we have $\bigtriangledown v_n \longrightarrow \bigtriangledown v$ in L^2. Hence $v_n \longrightarrow v$ in H^1_G.
$\square$

Proof of Theorem 3.2. The proof is a direct consequence of Lemmas 3.3, 3.4 and 3.5. $\square$

4 Proof of Theorem 1.1

To prove Theorem 1.1, note first that every critical point of the functional $\bar{J}_\epsilon$ corresponds to a weak solution of problem (2). Thus, we need to find a critical point for the functional $\bar{J}_\epsilon$. To do this, we shall show that the functionals $\bar{J}_\epsilon$ and $\bar{H}_\epsilon$ will coincide for the small values of ϵ. Hence, every critical point of $\bar{H}_\epsilon$ will be a critical point of $\bar{J}_\epsilon$. Also, it follows from Theorem 3.2 that $\bar{H}_\epsilon$ has a nontrivial critical point for every $\epsilon > 0$.

Without loss of generality, we may assume ϵ^2 instead of ϵ in the functionals $\bar{H}_\epsilon$ and $\bar{J}_\epsilon$, i.e.

$$\bar{H}_\epsilon(v) = \frac{\epsilon^2}{2} \int |\bigtriangledown v|^2 + \frac{1}{2} \int V(x) f(v)^2 dx - \int W(x, f(v))dx,$$

and

$$\bar{J}_\epsilon(v) = \frac{\epsilon^2}{2} \int_{\mathbb{R}^N} |\bigtriangledown v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f(v)^2 dx - \frac{1}{22^*} \int_{\mathbb{R}^N} |f(v)|^{22^*} dx.$$

It follows from Theorem 3.2 that there exists a critical point $v_\epsilon \in H_1^G$ of $\bar{H}_\epsilon(v)$ for each $\epsilon > 0$. Set $u_\epsilon = f(v_\epsilon)$.
The following Lemmas are crucial for the proof of Theorem 1.1.

Lemma 4.1. *The sequence $\{u_\epsilon\}_{\epsilon>0}$ is strongly convergent to 0 when $\epsilon \longrightarrow 0$, in $H^1(\mathbb{R}^N)$, i.e.*

$$\|u_\epsilon\|_{H^1} \longrightarrow 0 \quad as \quad \epsilon \longrightarrow 0.$$

Proof. Let $0 \not\equiv \phi \in C_{0,r}^{\infty}(\mathbb{R}^N)$ be a non-negative function with $\mathrm{supp}(\phi) \subset \Omega$ and $H_1(\phi) \leq 0$. Set $\gamma_1(t) := h(t\phi)$. Hence, we have

$$\bar{H}_\epsilon(\gamma_1(1)) = \bar{H}_\epsilon(h(\phi)) = H_\epsilon(\phi) \leq H_1(\phi) \leq 0.$$

It follows from the definition of the Mountain Pass value that

$$\bar{H}_\epsilon(v_\epsilon) = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \bar{H}_\epsilon(\gamma(t)) \leq \sup_{t\in[0,1]} \bar{H}_\epsilon(\gamma_1(t)) = \sup_{t\in[0,1]} \bar{H}_\epsilon(h(t\phi)) = \sup_{t\in[0,1]} H_\epsilon(t\phi).$$

Therefore, we obtain

$$\begin{aligned}
\bar{H}_\epsilon(v_\epsilon) &\leq \sup_{t\in[0,1]} H_\epsilon(t\phi) \\
&= \sup_{t\in[0,1]} \frac{\epsilon^2 t^2}{2}\int |\bigtriangledown \phi|^2 + \frac{\epsilon^2 t^4}{2}\int |\phi|^2|\bigtriangledown \phi|^2 - \frac{t^{22^*}}{22^*}\int |\phi|^{22^*}dx \\
&\leq \sup_{t\in[0,1]} \frac{\epsilon^2 t^2}{2}\int (1+|\phi|^2)|\bigtriangledown \phi|^2 dx - \frac{|t|^{22^*}}{22^*}\int |\phi|^{22^*}dx \\
&\leq (\frac{1}{2} - \frac{1}{22^*})\epsilon^{\frac{22^*}{2^*-1}} A(\phi) \qquad (32)
\end{aligned}$$

where $A(\phi) = \left(\frac{\int (1+|\phi|^2)|\bigtriangledown\phi|^2 dx}{\int |\phi|^{22^*}dx}\right)^{\frac{2^*}{2^*-1}}$. Now, as in the proof of part (i) of Lemma 3.5 we obtain

$$\begin{aligned}
\bar{H}_\epsilon(v_\epsilon) &= \bar{H}_\epsilon(v_\epsilon) - \frac{1}{\theta}\langle \bar{H}'(v_\epsilon), v_\epsilon\rangle \\
&\geq \epsilon^2(\frac{1}{2} - \frac{2}{\theta})\int |\bigtriangledown v_n|^2 dx + (\frac{1}{2} - \frac{1}{\theta})(1-\frac{1}{k})\int V(x) f(v_n)^2 dx. \qquad (33)
\end{aligned}$$

Combining (32) and (33), we get

$$\epsilon^2(\frac{1}{2} - \frac{2}{\theta})\int |\bigtriangledown v_n|^2 dx + (\frac{1}{2} - \frac{1}{\theta})(1-\frac{1}{k})\int V(x)|f(v_n)|^2 dx \leq (\frac{1}{2} - \frac{1}{22^*})\epsilon^{\frac{22^*}{2^*-1}} A(\phi).$$

Therefore

$$(\frac{1}{2} - \frac{2}{\theta})\int |\bigtriangledown v_n|^2 dx + (\frac{1}{2} - \frac{1}{\theta})(1-\frac{1}{k})\int V(x) f(v_n)^2 dx \leq (\frac{1}{2} - \frac{1}{22^*})\epsilon^{\frac{2}{2^*-1}} A(\phi). \qquad (34)$$

Hence, substituting $u_\epsilon = f(v_\epsilon)$ in (34) implies

$$\int (1+|u_\epsilon|^2)|\bigtriangledown u_\epsilon|^2 dx + \int V(x)|u_\epsilon|^2 dx \leq C\epsilon^{\frac{2}{2^*-1}} A(\phi).$$

Therefore

$$\|u_\epsilon\|_{H^1} \longrightarrow 0 \quad \text{as} \quad \epsilon \longrightarrow 0.$$

□

The following Lemma is standard (e.g [20]).

Lemma 4.2. *Let $N > 2$. There is a constant $C = C_N$, such that*

$$|u(x)| \leq \frac{C}{|x|^{\frac{N-2}{2}}} \|u\|_{H^1(\mathbb{R}^N)} \quad \forall x \neq 0,$$

for any $u \in H^1_r(\mathbb{R}^N)$.

Lemma 4.3. *For every compact set $Q \subset \mathbb{R}^N$ such that $0 \notin Q$, $\|u_\epsilon\|_{L^\infty(Q)} \longrightarrow 0$ as $\epsilon \longrightarrow 0$.*

Proof. For each $\epsilon > 0$, it follows from Lemma 4.2 that

$$0 \leq u_\epsilon(x) \leq \frac{C}{|x|^{\frac{N-2}{2}}} \|u_\epsilon\|_{H^1(\mathbb{R}^N)} \quad \forall x \neq 0,$$

which together with the result of Lemma 4.1 obviously means

$$\|u_\epsilon\|_{L^\infty(Q)} \longrightarrow 0 \quad \text{as} \quad \epsilon \longrightarrow 0.$$

□

Proof of Theorem 1.1. By Lemma 4.3 we have

$$M_\epsilon := \max_{x \in \bar{\Lambda}} f(v_\epsilon) \longrightarrow 0 \quad \text{as} \quad \epsilon \longrightarrow 0. \tag{35}$$

From (35) there exists $\epsilon_0 > 0$ such that $\max_{x \in \bar{\Lambda}} f(v_\epsilon) < \beta$ for every $0 < \epsilon < \epsilon_0$. Using the test function $\phi = \frac{(f(v_\epsilon)-\beta)_+}{f'(v_\epsilon)}$, we get

$$\begin{aligned} 0 = \langle \bar{H}'_\epsilon(v_\epsilon), \phi \rangle = \int_F \epsilon^2 (1 + \frac{f(v_\epsilon)^2}{1 + f(v_\epsilon)^2}) | \bigtriangledown v_\epsilon|^2 \quad &+ \quad \int_{\mathbb{R}^N \setminus \bar{\Lambda}} V(x) f(v_\epsilon)(f(v_\epsilon) - \beta)_+ dx \\ &- \quad \int_{\mathbb{R}^N \setminus \bar{\Lambda}} w(x, f(v_\epsilon))(f(v_\epsilon) - \beta)_+ dx \end{aligned}$$

where $F = (\mathbb{R}^N \setminus \bar{\Lambda}) \cap \{x | f(v_\epsilon) \geq \beta\}$. From (g_2), we have

$$V(x) f(v_\epsilon)(f(v_\epsilon) - \beta)_+ - w(x, f(v_\epsilon))(f(v_\epsilon) - \beta)_+ \geq 0, \quad \forall x \in \Lambda^c.$$

Thus,

$$\epsilon^2 \int_F (1 + \frac{f(v_\epsilon)^2}{1 + f(v_\epsilon)^2}) | \bigtriangledown v_\epsilon|^2 dx = 0,$$

from which we obtain

$$f(v_\epsilon) \leq \beta, \quad \forall x \in \mathbb{R}^N \setminus \bar{\Lambda}.$$

Therefore

$$w(x, f(v_\epsilon)) = f(v_\epsilon)^{22^*-1}, \quad \forall x \in \mathbb{R}^N \setminus \bar{\Lambda},$$

and we conclude that

$$\epsilon^2 \int \bigtriangledown v_\epsilon . \bigtriangledown \xi dx + \int V(x) f(v_\epsilon) f'(v_\epsilon) \xi dx = \int f(v_\epsilon)^{22^*-1} f'(v_\epsilon) \xi dx$$

for every $\xi \in H_G^1$ and $\epsilon \in (0, \epsilon_0)$. Therefore, $\bar{J}_\epsilon(v)$ has a critical point v_ϵ in H_G^1 for every $\epsilon \in (0, \epsilon_0)$. □

Remark 4.4. Note that, as in the argument in the proof of Theorem 1.1, it seems the smallness of ϵ is required for technical reasons. In fact, the smallness of ϵ ensures that the deformed functional H_ϵ and the main functional J_ϵ coincide and in result they have the same critical points. However, we don't know if solutions exist for large values of ϵ. Indeed, even for the semilinear case ($k = 0$),i.e.

$$-\epsilon \Delta u + V(x) u = \mu |u|^{p-1} u, \quad u > 0, x \in \mathbb{R}^N, p + 1 = 2^*, \tag{36}$$

the existence of solutions depends on the graph topology of coefficient $V(x)$ and the smallness of ϵ. In fact, even for this simpler case, it is not quite clear if solutions exist for large values of ϵ.

Acknowledgment: The author is grateful to the referee for his/her critical comments which improved the original manuscript.

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